A Short Path Joining Two Zeros Inside a Polynomial Lemniscate

Abstract

Let

$$f(z) = \prod_{j=1}^{n} (z - z_j)$$

be a monic polynomial whose zeros are listed with multiplicity and satisfy $|z_j| < 1$ for all j. Assuming the known Erdős–Herzog–Piranian component lemma that some connected component of

$$\Lambda_f := \{z \in \mathbb{C} : |f(z)| < 1\}$$

contains at least two zeros of f, we prove that there exist two roots of f which can be joined by a rectifiable curve of Euclidean length < 2 contained in Λ_f. The proof combines a sharp integral estimate on each component of a polynomial lemniscate, an explicit two-parameter perturbation that makes the critical-level structure above the first critical value Morse and excellent on a fixed collar, and a spanning-tree construction in the resulting Reeb decomposition.

1 Introduction

Let

$$f(z) = \prod_{j=1}^{n} (z - z_j), \qquad |z_j| < 1 \qquad (1 \le j \le n),$$

where the zeros are listed with multiplicity. We consider the filled unit lemniscate

$$\Lambda_f := \{z \in \mathbb{C} : |f(z)| < 1\}.$$

The question addressed here is the following.

Must there always exist a rectifiable curve of Euclidean length less than 2, contained in Λ_f, whose endpoints are two roots of f?

Our main result answers this in the affirmative.

Theorem 1. *Let*

$$f(z) = \prod_{j=1}^{n} (z - z_j), \qquad |z_j| < 1 \qquad (1 \le j \le n),$$

where the zeros are listed with multiplicity. Then there exist indices $i, j \in \{1, \dots, n\}$ and a rectifiable curve $\gamma \subset \Lambda_f$ joining z_i to z_j such that

$$\operatorname{len}(\gamma) < 2.$$

In particular, if f is squarefree, the endpoints are distinct zeros of f.

Remark 2. If f has a multiple zero, the conclusion is immediate: two coincident roots may be joined by the constant curve. Thus the only interesting case is the squarefree one. In the proof below we therefore reduce immediately to squarefree polynomials.

The only external input we use is the following component lemma.

Component lemma. Some connected component of Λ_f contains at least two zeros of f.

This is now known by recent work of Ghosh and Ramachandran, answering a problem posed in 1958 by Erdős, Herzog, and Piranian; see [1, 2, 3].

The proof of Theorem 1 has three parts.

1. For any component $U \subset \Lambda_f$ containing m zeros of f, we prove the sharp integral estimate
$$\int_U \frac{|f'(z)|}{|f(z)|}\, dA(z) \le 2\pi m.$$
2. Above the first critical level of $u := -\log|f|$, we construct in a generic perturbation a finite tree spanning the relevant roots, with total length strictly less than m.
3. A small explicit perturbation of the form $f \mapsto f + \lambda z + \beta$ makes the critical-point configuration generic on a fixed collar, while preserving the relevant component and a fixed positive slack below the threshold 2.

2 An analytic estimate on a component

Let U be a connected component of Λ_f, and suppose that U contains exactly $m \ge 1$ zeros of f, counted with multiplicity. Set
$$u := -\log|f| \quad \text{on } U.$$
Then $u > 0$ on U, $u = 0$ on ∂U, and $u(z) \to +\infty$ at the zeros of f in U. For $t \ge 0$ define
$$K_t := \{z \in U : u(z) \ge t\}, \qquad A(t) := \operatorname{Area}(K_t).$$
For regular t write
$$P(t) := \mathcal{H}^1\big(\{z \in U : u(z) = t\}\big).$$
We also set
$$I(U) := \int_U \frac{|f'(z)|}{|f(z)|}\, dA(z) = \int_U |\nabla u|\, dA.$$

Proposition 3. *With the notation above,*
$$I(U) \le 2\pi m.$$

Proof. By the coarea formula,
$$I(U) = \int_0^\infty P(t)\, dt. \tag{1}$$
Also, for regular t,
$$-A'(t) = \int_{u=t} \frac{ds}{|\nabla u|}. \tag{2}$$
Since
$$\Delta u = -2\pi \sum_{a \in Z(f)\cap U} m_a\, \delta_a$$
in the sense of distributions, Green's formula on K_t gives, for every regular t,
$$\int_{u=t} |\nabla u|\, ds = 2\pi m. \tag{3}$$

Hence Cauchy–Schwarz yields

$$P(t)^2 \leq \left(\int_{u=t} |\nabla u|\, ds\right)\left(\int_{u=t} \frac{ds}{|\nabla u|}\right) = 2\pi m\,(-A'(t)). \tag{4}$$

We next estimate $A(t)$. Consider the condenser (U, K_t). The extremal function is

$$v_t(z) := \min\left(\frac{u(z)}{t}, 1\right),$$

so, with the normalization

$$\operatorname{cap}(U, K_t) = \frac{1}{4\pi} \inf \int |\nabla v|^2\, dA,$$

one has

$$\operatorname{cap}(U, K_t) = \frac{1}{4\pi t^2} \int_{0<u<t} |\nabla u|^2\, dA.$$

Using coarea and (3),

$$\int_{0<u<t} |\nabla u|^2\, dA = \int_0^t \left(\int_{u=s} |\nabla u|\, ds\right) ds = \int_0^t 2\pi m\, ds = 2\pi m t.$$

Therefore

$$\operatorname{cap}(U, K_t) = \frac{m}{2t}. \tag{5}$$

We now invoke two classical inequalities from planar potential theory.

(i) Carleman's condenser inequality:

$$\frac{1}{\operatorname{cap}(E, B)} \leq \log \frac{\operatorname{Area}(E)}{\operatorname{Area}(B)}.$$

(ii) Pólya's area-capacity inequality:

$$\operatorname{Area}(K) \leq \pi \ \operatorname{cap}(K)^2$$

for compact planar sets K.

These are summarized, with the normalization used here, in [1]. Applying Carleman's inequality to (U, K_t) and using (5), we obtain

$$\log \frac{\operatorname{Area}(U)}{A(t)} \leq \frac{2t}{m},$$

that is,

$$A(t) \leq \operatorname{Area}(U) e^{-2t/m}. \tag{6}$$

Since $U \subset f^{-1}(\mathbb{D})$ and f is monic, the logarithmic capacity of U is at most 1; therefore Pólya's inequality gives

$$\operatorname{Area}(U) \leq \pi. \tag{7}$$

Combining (6) and (7),

$$A(t) \leq \pi e^{-2t/m}. \tag{8}$$

Now from (4),

$$P(t) \leq \sqrt{2\pi m\,(-A'(t))}.$$

Using the elementary inequality $2\sqrt{xy} \le x/\lambda + \lambda y$ with $x = 2\pi m$, $y = -A'(t)$, and $\lambda = me^{t/m}$, we get

$$P(t) \le \pi e^{-t/m} + \frac{m}{2} e^{t/m}(-A'(t)).$$

Integrating and using (1),

$$I(U) \le \pi \int_0^\infty e^{-t/m}\, dt + \frac{m}{2} \int_0^\infty e^{t/m}(-A'(t))\, dt.$$

The first integral equals πm. For the second, integration by parts gives

$$\int_0^\infty e^{t/m}(-A'(t))\, dt = A(0) + \frac{1}{m} \int_0^\infty e^{t/m} A(t)\, dt.$$

Using (7) and (8),

$$A(0) \le \pi, \qquad \frac{1}{m} \int_0^\infty e^{t/m} A(t)\, dt \le \frac{\pi}{m} \int_0^\infty e^{-t/m}\, dt = \pi.$$

Therefore

$$\int_0^\infty e^{t/m}(-A'(t))\, dt \le 2\pi,$$

and hence

$$I(U) \le \pi m + \frac{m}{2}(2\pi) = 2\pi m.$$

□

Corollary 4. *Let $c \ge 0$, and let W be a connected component of $\{z \in U : u(z) > c\}$. If W contains exactly M zeros of f, counted with multiplicity, then*

$$\int_W |\nabla u|\, dA \le 2\pi M.$$

Equivalently,

$$\frac{1}{2\pi} \int_c^\infty P_W(t)\, dt \le M,$$

where $P_W(t) = \mathcal{H}^1(\{z \in W : u(z) = t\})$ for regular $t > c$.

Proof. Set $h := e^c f$ and $v := -\log|h| = u - c$. Then W is a connected component of $\{v > 0\} = \{|h| < 1\}$, and W contains exactly M zeros of h.

The proof of Proposition 3 applies verbatim to h on W. The only place where monicity was used was in the bound $\mathrm{Area}(W) \le \pi$, obtained via Pólya's inequality. Here the leading coefficient of h is e^c, so the filled lemniscate $\{|h| \le 1\}$ has logarithmic capacity $e^{-c/n} \le 1$; by monotonicity, $\mathrm{cap}(W) \le 1$, and again Pólya gives $\mathrm{Area}(W) \le \pi$. Therefore the same argument yields

$$\int_W |\nabla v|\, dA \le 2\pi M.$$

Since $\nabla v = \nabla u$, this is exactly the desired estimate. □

3 The first critical value and the low superlevel sets

Throughout the rest of the paper, f is assumed squarefree on the component under consideration. Let $U \subset \Lambda_f$ be a connected component containing exactly $m \geq 2$ zeros of f, and write again

$$u = -\log|f| \quad \text{on } U.$$

Lemma 5. *The function u has at least one critical point in U. Consequently the set of positive critical values of u is nonempty, and it has a smallest element. Denote that smallest positive critical value by λ_*.*

Proof. Choose pairwise disjoint closed disks $E_1, \ldots, E_m \Subset U$ centered at the zeros of f. Since the superlevel sets of u shrink to the zeros of f as the level tends to $+\infty$, there exists $T > 0$ such that

$$K_T := \{u \geq T\} \subset \bigcup_{j=1}^{m} \operatorname{int}(E_j).$$

In particular $u < T$ on $\bigcup_j \partial E_j$, and K_T has at least m connected components, one contained in each E_j.

Assume for contradiction that u has no critical point in $\{0 < u \leq T\}$. Then

$$u : \{0 < u \leq T\} \to (0, T]$$

is a proper submersion. Hence this region is a product collar over any regular level $\{u = t\}$; equivalently, for every $0 < t \leq T$, the set $U \setminus K_t$ is diffeomorphic to $\{u = t\} \times (0, t)$. It follows that U and K_t have the same number of connected components for every $0 < t \leq T$. Since $U = K_0$ is connected, K_T must be connected, contradicting the previous paragraph. Therefore u has a critical point in $\{0 < u \leq T\}$, and the set of positive critical values is nonempty.

Since u has only finitely many critical points in U (they are zeros of f' and f is squarefree on U), the set of positive critical values is finite. Hence it has a smallest element, denoted λ_*. ☐

Set

$$\alpha := \lambda_*/4.$$

Then u has no critical values in $[0, 4\alpha)$.

Lemma 6. *For every $t \in [0, \lambda_*)$, the superlevel set $K_t := \{u \geq t\}$ is connected. In particular,*

$$K := \{u \geq \alpha/2\}, \qquad L := \{u \geq 3\alpha/2\}$$

are compact and connected.

Proof. Fix $0 < t < \lambda_*$. By definition of λ_*, the function u has no critical points in $\{0 < u \leq t\}$. As in the proof of Lemma 5, the map $u : \{0 < u \leq t\} \to (0, t]$ is a proper submersion, so U and K_t have the same number of connected components. Since U is connected, K_t is connected for every $0 < t < \lambda_*$. The case $t = 0$ is immediate. Compactness of K and L follows because they are closed subsets of the bounded domain U. ☐

Define

$$q := \frac{1}{2\pi} \int_{\alpha}^{2\alpha} P(t)\, dt.$$

Since $P(t) > 0$ for $\alpha < t < 2\alpha$, we have $q > 0$. By Corollary 4,

$$\frac{1}{2\pi} \int_{2\alpha}^{\infty} P(t)\, dt \leq m - q. \tag{9}$$

4 A generic spanning tree above the first critical level

We now show that, under a genericity assumption on the critical points above level 2α, one can realize the tail budget as the length of a connected spanning tree up to an arbitrarily small error.

Proposition 7. *Assume that:*

(a) all critical points of u with value $> 2\alpha$ are nondegenerate;

(b) all critical values of u larger than 2α are pairwise distinct.

Then for every $\varepsilon > 0$ there exists a finite embedded tree $G_\varepsilon \subset U$ containing all m zeros of f in U and satisfying

$$\operatorname{len}(G_\varepsilon) \le \frac{1}{2\pi}\int_{2\alpha}^{\infty} P(t)\,dt + \varepsilon. \tag{10}$$

Consequently,

$$\operatorname{len}(G_\varepsilon) \le m - q + \varepsilon. \tag{11}$$

Proof. Let the distinct critical values of u above 2α be

$$2\alpha < \mu_1 < \mu_2 < \cdots < \mu_s.$$

Choose pairwise disjoint closed intervals

$$I_r = [\mu_r - \delta_r, \mu_r + \delta_r]$$

so small that each I_r contains no critical values of u other than μ_r. Next choose a regular value $T > \mu_s$ so large that the components $Z_1, \dots, Z_m$ of $\{u > T\}$ have pairwise disjoint closures and are all contained in arbitrarily small neighborhoods of the zeros of f. This is possible because f is squarefree on U, so near each zero a_j one has

$$u(z) = -\log|z - a_j| + O(1),$$

and therefore the components of $\{u > T\}$ shrink to the zeros as $T \to \infty$.

Set

$$R := \{2\alpha < u < T\} \setminus u^{-1}\left(\bigcup_{r=1}^{s} I_r\right).$$

Each connected component S of R is a regular strip: there exist regular values $a_S < b_S$ with

$$2\alpha \le a_S < b_S \le T$$

such that $u(S) = (a_S, b_S)$, the function u has no critical points in S, and for each $t \in (a_S, b_S)$ the set

$$\Gamma_t := \{u = t\} \cap S$$

is carried diffeomorphically to every other $\Gamma_{t'}$ by the flow of

$$X := \frac{\nabla u}{|\nabla u|^2}.$$

The lower endpoint a_S is either 2α or one of the regular values $\mu_r + \delta_r$, and the upper endpoint b_S is either T or one of the regular values $\mu_r - \delta_r$.

Let k_S be the number of zeros of f contained in the connected component of $\{u > t\}$ that lies above S; this is independent of the regular value $t \in (a_S, b_S)$. For $t \in (a_S, b_S)$ the flux identity gives

$$\int_{\Gamma_t} |\nabla u|\, ds = 2\pi k_S.$$

Thus

$$d\mu_t := \frac{|\nabla u|\, ds}{2\pi k_S}$$

is a probability measure on Γ_t. We claim that the family (μ_t) is preserved by the flow of X. Indeed, let $a_S < s < t < b_S$, let $\Phi_{s,t} : \Gamma_s \to \Gamma_t$ be the flow map, and let $E \subset \Gamma_s$ be measurable. The union of the flow lines issued from E between times s and t is a curvilinear tube $T(E; s, t) \subset S$ whose boundary consists of E, its image $\Phi_{s,t}(E)$, and two flow lines. Since u is harmonic on S, Green's formula gives

$$0 = \int_{\partial T(E;s,t)} \partial_\nu u\, ds.$$

On the side boundaries, ∇u is tangent to the flow lines of X, so $\partial_\nu u = 0$. On the cross-sections E and $\Phi_{s,t}(E)$, the outward normal is $\pm \nabla u/|\nabla u|$, hence

$$\int_E |\nabla u|\, ds = \int_{\Phi_{s,t}(E)} |\nabla u|\, ds.$$

Therefore $(\Phi_{s,t})_{\#}\mu_s = \mu_t$.

Fix once and for all a regular value $s \in (a_S, b_S)$. For $x \in \Gamma_s$, let γ_x be the trajectory segment in S obtained by following the flow of X from level a_S to level b_S. Since u increases at unit speed along X, its length is

$$\operatorname{len}(\gamma_x) = \int_{a_S}^{b_S} \frac{dt}{|\nabla u(\Phi_{s,t}(x))|}.$$

Averaging with respect to μ_s and using the measure invariance just proved, we obtain

$$\begin{aligned}
\int_{\Gamma_s} \operatorname{len}(\gamma_x)\, d\mu_s(x) &= \int_{a_S}^{b_S} \left(\int_{\Gamma_s} \frac{d\mu_s(x)}{|\nabla u(\Phi_{s,t}(x))|} \right) dt \\
&= \int_{a_S}^{b_S} \left(\int_{\Gamma_t} \frac{d\mu_t(y)}{|\nabla u(y)|} \right) dt \\
&= \frac{1}{2\pi k_S} \int_{a_S}^{b_S} \left(\int_{\Gamma_t} ds \right) dt = \frac{1}{2\pi k_S} \int_{a_S}^{b_S} P_S(t)\, dt,
\end{aligned}$$

where $P_S(t) := \mathcal{H}^1(\Gamma_t)$. Consequently there exists a trajectory $\gamma_S \subset S$ such that

$$\operatorname{len}(\gamma_S) \le \frac{1}{2\pi k_S} \int_{a_S}^{b_S} P_S(t)\, dt. \tag{12}$$

Summing over all strips and using $k_S \ge 1$,

$$\sum_S \operatorname{len}(\gamma_S) \le \frac{1}{2\pi} \int_{(2\alpha,T)\setminus \bigcup I_r} P(t)\, dt \le \frac{1}{2\pi} \int_{2\alpha}^{\infty} P(t)\, dt. \tag{13}$$

We now reconnect these strip segments near the critical points and inside the truncated top components Z_j. For each critical point p_r of u with value μ_r, choose a closed neighborhood N_r so small that the sets N_r are pairwise disjoint, each incident strip meets ∂N_r in exactly one short cross-section, and inside N_r all the corresponding boundary points together with p_r can be connected by an embedded tree $T_r \subset N_r$ of length $< \eta_r$, where

$$\sum_{r=1}^{s} \eta_r < \frac{\varepsilon}{2}.$$

This is possible by the Morse normal form

$$u = u(p_r) + x^2 - y^2$$

near each nondegenerate critical point; see, for example, [4].

For each zero a_j of f in U, there is exactly one strip whose upper boundary lies on ∂Z_j; let b_j be the endpoint of the chosen trajectory on that boundary. Because u has no critical points on $Z_j \setminus \{a_j\}$ and Z_j shrinks to a_j as $T \to \infty$, the point b_j can be joined to a_j by a curve $\sigma_j \subset Z_j$ of arbitrarily small length. Enlarging T if necessary, we may assume

$$\sum_{j=1}^{m} \operatorname{len}(\sigma_j) < \frac{\varepsilon}{2}.$$

Let

$$H := \left(\bigcup_{S} \gamma_S\right) \cup \left(\bigcup_{r=1}^{s} T_r\right) \cup \left(\bigcup_{j=1}^{m} \sigma_j\right).$$

By construction, the incidence graph of H is the finite tree obtained from the Reeb tree of the superlevel decomposition above level 2α by truncating its m top leaves at level T and then reattaching the corresponding zeros inside the sets Z_j. In particular H is connected and contains all m zeros of f in U. Its total length is at most

$$\frac{1}{2\pi}\int_{2\alpha}^{\infty} P(t)\,dt + \varepsilon.$$

Finally, pruning cycles does not increase total length, so we may assume the resulting connected graph is a tree. This proves (10), and (11) follows from (9). ☐

5 An explicit perturbation lemma

We now replace the abstract transversality step by a concrete two-parameter perturbation.

Fix a squarefree monic polynomial f, a component $U \subset \Lambda_f$ containing exactly $m \geq 2$ zeros of f, and the number α introduced above. Define the compact collars

$$K := \{z \in U : u(z) \geq \alpha/2\}, \qquad L := \{z \in U : u(z) \geq 3\alpha/2\}.$$

Then K and L are compact and connected by Lemma 6.

Choose pairwise disjoint closed disks

$$D_1, \ldots, D_m \subset \operatorname{int}(L)$$

centered at the zeros of f in U, so small that each D_j contains exactly one zero and

$$L_0 := L \setminus \bigcup_{j=1}^{m} \operatorname{int}(D_j)$$

is connected. This is possible because L is a connected planar domain and finitely many sufficiently small pairwise disjoint closed disks compactly contained in L do not disconnect it.

For $\lambda, \beta \in \mathbb{C}$ define

$$g_{\lambda,\beta}(z) := f(z) + \lambda z + \beta, \qquad u_{\lambda,\beta} := -\log |g_{\lambda,\beta}|.$$

Lemma 8 (Explicit excellent perturbation)**.** *For every $\eta_0 > 0$ there exist $\lambda, \beta \in \mathbb{C}$ with $|\lambda| + |\beta| < \eta_0$ such that $g := g_{\lambda,\beta}$ satisfies:*

(1) each disk D_j contains exactly one zero of g, and g has no other zeros in K;

(2) no critical point of u_g lying in K has critical value in $[\alpha, 2\alpha]$;

(3) every critical point of u_g in K with critical value $> 2\alpha$ is nondegenerate;

(4) the critical values of u_g arising from critical points in K and larger than 2α are pairwise distinct.

Proof. We proceed in five steps.

Step 1: preserve the relevant zeros. Choose $\eta_0 > 0$ so small that on every ∂D_j and on $K \setminus \bigcup_j D_j$ one has

$$|\lambda z + \beta| < |f(z)|$$

whenever $|\lambda| + |\beta| < \eta_0$. By Rouché's theorem, for every such pair (λ, β) the polynomial $g_{\lambda,\beta}$ has exactly one zero in each D_j and no zeros in $K \setminus \bigcup_j D_j$.

Step 2: make all critical points simple. Since

$$g'_{\lambda,\beta}(z) = f'(z) + \lambda,$$

the critical points of u_g away from the zeros of g are precisely the zeros of $f' + \lambda$. A zero of $f' + \lambda$ is multiple if and only if it is also a zero of f''. Therefore the bad set

$$\Sigma_1 := \{-f'(\zeta) : f''(\zeta) = 0\}$$

is finite, and for every $\lambda \notin \Sigma_1$ all zeros of $f' + \lambda$ are simple. This implies that every critical point of u_g is nondegenerate.

Step 3: make the complex critical values distinct. Set

$$H_\lambda(z) := f(z) + \lambda z.$$

Its critical points are the zeros of $H'_\lambda(z) = f'(z) + \lambda$. The critical values of H_λ are exactly the roots of the resultant polynomial

$$R_\lambda(w) := \operatorname{Res}_z(f'(z) + \lambda, f(z) + \lambda z - w).$$

Two critical values coincide if and only if the discriminant

$$\Delta(\lambda) := \operatorname{Disc}_w R_\lambda(w)$$

vanishes. Thus the bad set

$$\Sigma_2 := \{\lambda : \Delta(\lambda) = 0\}$$

is algebraic. It is not all of $\mathbb{C}$: for large $|\lambda|$, the critical points of H_λ satisfy $f'(z) + \lambda = 0$, so if $\rho^{n-1} = -\lambda/n$, then

$$c_k(\lambda) = \rho\,\omega_k + O(1), \qquad \omega_k^{n-1} = 1,$$

and hence

$$H_\lambda(c_k(\lambda)) = (1-n)\rho^n\omega_k + O(|\rho|^{n-1}).$$

These values are pairwise distinct for large $|\lambda|$, so $\Delta \not\equiv 0$. Since Δ is a polynomial, Σ_2 is finite. Choose a small $\lambda \notin \Sigma_1 \cup \Sigma_2$. Then H_λ has simple critical points and pairwise distinct complex critical values.

Step 4: keep all critical values in K above 2α. Because $u = -\log|f|$ has no critical values in $[0, 4\alpha)$, every critical point c of f in K satisfies

$$|f(c)| \le e^{-4\alpha}.$$

Since f' has no zeros on ∂K, for all sufficiently small λ the zeros of $f' + \lambda$ in K remain close to the zeros of f' in K, and therefore every critical point c of H_λ in K satisfies

$$|H_\lambda(c)| < e^{-3\alpha}.$$

Now choose β so small that

$$|\beta| < e^{-2\alpha} - e^{-3\alpha}.$$

Then for every critical point c of H_λ in K,

$$|g(c)| = |H_\lambda(c) + \beta| < e^{-2\alpha}.$$

Equivalently, every critical value of u_g arising from a critical point in K is $> 2\alpha$; in particular no critical point of u_g in K has critical value in $[\alpha, 2\alpha]$.

Step 5: make the moduli distinct. Let $a_1, \ldots, a_N$ be the pairwise distinct complex critical values of H_λ arising from critical points in K. After adding β, they become $a_1 + \beta, \ldots, a_N + \beta$. Their moduli fail to be distinct only if, for some $i \neq j$,

$$|a_i + \beta| = |a_j + \beta|.$$

For fixed $i \neq j$, this equation describes an affine real line in the β-plane. Thus the bad values of β lie in a finite union of lines. This union has empty interior, so we may choose β arbitrarily small, outside it, while preserving the smallness required in Steps 1 and 4. Then the critical values of u_g arising from critical points in K and larger than 2α are pairwise distinct.

This proves the lemma. □

6 Stability of the collar integral and of the relevant component

We retain the notation of the previous section.

Lemma 9 (Component stability). *Suppose $g \to f$ uniformly on K and that, for all sufficiently close g,*

(a) each D_j contains exactly one zero of g;

(b) g has no zeros on $K \setminus \bigcup_j D_j$.

Then, for all sufficiently small perturbations, there exists a unique connected component V_g of

$$\{z \in K : -\log|g(z)| > \alpha\}$$

such that

$$L_0 \subset V_g \subset \operatorname{int}(K) \subset U,$$

and V_g contains exactly the m zeros of g lying in the disks D_j.

Proof. On L_0 one has $u \geq 3\alpha/2$, while on ∂K one has $u = \alpha/2$. Uniform convergence implies

$$-\log|g| > \alpha \quad \text{on } L_0, \qquad -\log|g| < \alpha \quad \text{on } \partial K$$

for all sufficiently small perturbations. Thus the connected set L_0 lies in a unique component

$$V_g \subset \operatorname{int}(K)$$

of $\{-\log|g| > \alpha\} \cap K$.

Fix j. Since $\partial D_j \subset L_0 \subset V_g$, we have $|g| < e^{-\alpha}$ on ∂D_j. By the maximum modulus principle,

$$|g| < e^{-\alpha} \qquad \text{throughout } D_j.$$

Hence $D_j \subset V_g$, so the zero of g inside D_j belongs to V_g. There are no other zeros of g in K, hence V_g contains exactly these m zeros.

Finally, if W were another component of $\{-\log|g| > \alpha\} \cap K$, then $W \Subset K \setminus \bigcup_j D_j$, so g would have no zeros on W and $-\log|g|$ would be harmonic there. Since $-\log|g| = \alpha$ on ∂W, the maximum principle would force $-\log|g| \equiv \alpha$ on W, a contradiction. Therefore V_g is the unique component of $\{-\log|g| > \alpha\} \cap K$. ☐

Lemma 10 (Continuity of the collar integral)**.** *Assume $g \to f$ in C^1 on the compact band*

$$B := \{z \in U : \alpha/2 \leq u(z) \leq 5\alpha/2\},$$

and assume that g has no zeros on B. Let V_g be the component from Lemma 9, and define

$$q_g := \frac{1}{2\pi}\int_{\alpha}^{2\alpha} P_g(t)\,dt,$$

where $P_g(t)$ denotes the total length of $\{z \in V_g : -\log|g(z)| = t\}$. Then

$$q_g \to q.$$

In particular, for all sufficiently small perturbations,

$$q_g > q/2. \tag{14}$$

Proof. Since u has no critical points on B, there exists $c > 0$ with $|\nabla u| \geq c$ on B. By C^1 convergence, for g sufficiently close to f one has $|\nabla(-\log|g|)| \geq c/2$ on B, so there are no critical points in the collar.

By Lemma 9, V_g is the unique component of $\{-\log|g| > \alpha\} \cap K$. If $z \in K$ satisfies $\alpha < -\log|g(z)| < 2\alpha$, then z belongs to $\{-\log|g| > \alpha\} \cap K$, hence necessarily to V_g. Thus

$$\{z \in V_g : \alpha < -\log|g(z)| < 2\alpha\} = \{z \in K : \alpha < -\log|g(z)| < 2\alpha\}.$$

Therefore, by coarea,

$$2\pi q = \int_{\{\alpha < u < 2\alpha\}} |\nabla u|\,dA, \qquad 2\pi q_g = \int_{\{z \in K : \alpha < -\log|g(z)| < 2\alpha\}} |\nabla(-\log|g|)|\,dA.$$

The characteristic functions of the corresponding collar regions converge almost everywhere in B, and the gradients converge uniformly on B. Dominated convergence therefore gives $q_g \to q$. ☐

7 Proof of the main theorem

We now complete the argument.

Lemma 11 (A tree lemma)**.** *Let T be a finite tree of total length L, and let $x_1, \ldots, x_M$ be marked points of T. Then some pair x_i, x_j is joined by a subpath of length at most $2L/M$.*

Proof. Subdivide edges at the marked points; this does not change the total length and turns the marked points into marked vertices. Take the minimal subtree spanning the marked vertices. Every leaf of this subtree is marked. Doubling every edge produces an Eulerian graph of total length $2L$. Traversing an Euler tour and recording the first appearances of the marked vertices gives a cyclic ordering of the M marked vertices. The M gaps between consecutive marked vertices along the tour have total length $2L$, so one of them has length at most $2L/M$. The corresponding unique path in the original tree has at most that length. ☐

Proof of Theorem 1. If f has a multiple zero, the conclusion is immediate, so we may assume that f is squarefree.

By the component lemma, there exists a connected component $U \subset \Lambda_f$ containing exactly $m \geq 2$ zeros of f. Let $u = -\log|f|$ on U, let α and q be as in Section 3, and choose the root disks $D_1, \ldots, D_m$ as in Section 5, so small that

$$\sum_{j=1}^{m} \operatorname{diam}(D_j) < \frac{q}{4m}. \tag{15}$$

Let $g = g_{\lambda,\beta}$ be an explicit perturbation provided by Lemma 8, chosen so small that, in addition,

$$g \to f \qquad \text{in } C^1(B),$$

where B is the collar from Lemma 10. Since f has a positive lower bound on B, this also guarantees that g has no zeros on B. Let V_g be the corresponding component from Lemma 9. Then $V_g \subset U$, it contains exactly the m perturbed roots $b_1, \ldots, b_m$ lying in the disks D_j, and by Lemma 10 we have $q_g > q/2$.

Applying Corollary 4 to the monic polynomial g on the component V_g with the shift parameter $c = \alpha$, and then subtracting the collar integral, gives

$$\frac{1}{2\pi} \int_{2\alpha}^{\infty} P_g(t)\, dt \leq m - q_g < m - \frac{q}{2}.$$

Now Proposition 7 applies to $u_g = -\log|g|$ on V_g: indeed, Lemma 8(3)–(4) give the required Morse and distinct-critical-value hypotheses for all critical points of u_g in V_g with value $> 2\alpha$, because $V_g \subset K$. Moreover, V_g is connected, and Lemma 8(2) shows that u_g has no critical values in $[\alpha, 2\alpha]$ on K. Hence, exactly as in Lemma 6, the superlevel sets $\{z \in V_g : u_g(z) \geq t\}$ are connected for $t \in [\alpha, 2\alpha]$; in particular, the level 2α is a connected base level for the construction in Proposition 7. Choosing $\varepsilon = q/8$, we obtain a finite tree $G \subset V_g \subset U$ containing the m perturbed roots, with

$$\operatorname{len}(G) \leq m - q_g + \frac{q}{8} < m - \frac{3q}{8}.$$

By Lemma 11, some pair of perturbed roots, say $b_r \in D_r$ and $b_s \in D_s$, is joined in G by a path of length

$$\frac{2}{m}\left(m - \frac{3q}{8}\right) = 2 - \frac{3q}{4m}. \tag{16}$$

Since each D_j contains exactly one zero of g, the indices r and s correspond to distinct original zeros of f. Joining b_r to the original zero in D_r and b_s to the original zero in D_s by straight segments inside those disks adds total length less than $q/(4m)$ by (15). Combining with (16), we obtain a curve in $U \subset \Lambda_f$ joining two zeros of f and having total length

$$\left(2 - \frac{3q}{4m}\right) + \frac{q}{4m} = 2 - \frac{q}{2m} < 2.$$

This proves the theorem. $\square$

References

[1] E. Crane, *The area of polynomial images and preimages*, Bull. Lond. Math. Soc. **36** (2004), 786–792.

[2] P. Erdős, F. Herzog, and G. Piranian, *Metric properties of polynomials*, J. Analyse Math. **6** (1958), 123–148.

[3] S. Ghosh and K. Ramachandran, *Number of components of polynomial lemniscates: a problem of Erdős, Herzog, and Piranian*, J. Math. Anal. Appl. **540** (2024), no. 1, 128571.

[4] J. Milnor, *Morse Theory*, Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, NJ, 1963.